THE SENSITIVITY OF GRANGER CAUSALITY

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Overview

1. **Introduction**

2. Characterisations of Granger Causality

3. Granger Causality and Additive Noise

Introduction

Causality

- often known a priori, in particular in physics. E.g. Ohm's law $U = RI$. This formula does not reveal the direction of causation
- causal inference: the process of drawing a conclusion about a causal connection

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2011 has been awarded to T.J. Sargent and C. Sims "for their empirical research on cause and effect in the macroeconomy"

Granger Causality

Let $(X(t) \mid t \in \mathbb{Z}), X(t) : \Omega \to \mathbb{R}^d$ be a weakly stationary, centered and Gaussian vector process with full rank stationary rational spectral density $f(z)$.
Write

$$X(t) = \begin{pmatrix} X_A(t) \\ X_B(t) \end{pmatrix}.$$

We say that $(X_A(t))$ does not **Granger cause** (GC) $(X_B(t))$ if

$$\mathrm{E}(X_B(t) \mid X_A(s), X_B(s), s < t) = \mathrm{E}(X_B(s) \mid X_B(s), s < t).$$

Instantaneous causality

$$\mathrm{E}(X_B(t) \mid X_A(t), X_A(s), X_B(s), s < t) \neq \mathrm{E}(X_B(t) \mid X_A(s), X_B(s), s < t).$$

Post hoc, ergo propter hoc Fallacy.

Other concepts:

- Peter Bühlmann (2018): Invariance, Causality and Robustness. *ETH Zürich*.
- Jim Heckman (2014): Causal Models, Structural Equations and Identification. *Asia Meeting of The Econometric Society*, Taipei.
- Judea Pearle (2000): Causality: Models, Reasoning and Inference. *Cambridge University Press*.

Questions of this Presentation

How sensitive is GC with respect to

- errors in variables
- (linear transformations)
- (subsampling)

Anderson, B.D.O., M. Deistler and J.B. Dufour (2019): On the Sensitivity of Granger Causality to Errors-in-Variables, Linear Transformations and Subsampling. *Journal of Time Series Analysis*, 40, 102-123.

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Characterizations of Granger Causality

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Characterizations of Granger Causality

Wold representation (based on spectral factorization)

$$f(z) = W(z)QW^*(z) = \begin{pmatrix} f_{AA}(z) & f_{AB}(z) \\ f_{BA}(z) & f_{BB}(z) \end{pmatrix}$$

$W(z)$ square, real rational, stable and (strictly) miniphase transferfunction, $W(0) = I, Q > 0$.

$$X(t) = W(z)\varepsilon(t); \quad W(z) = \begin{pmatrix} W_{11}(z) & W_{12}(z) \\ W_{11}(z) & W_{22}(z) \end{pmatrix}; \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

$AR(\infty)$ representation of $(X(t))$:

$$\pi(z)X(t) = \varepsilon(t); \quad X(t) = \underbrace{a(z)X(t)}_{\sum_{i=1}^{\infty} A_i X(t-i)} + \varepsilon(t)$$

$$a(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

$(\varepsilon(t))$ are innovations,

Characterizations of Granger Causality

Theorem 1 (Canonical spectral factor characterizaion)
The following conditions are equivalent:

(i) X_A does not GC X_B

(ii) $W_{21}(z) = 0$.

The following conditions are also equivalent:

(i) X_A neither GC X_B, nor does it cause X_B instantaneously

(ii) $W_{21}(z) = 0$ and Q is block diagonal.

Theorem 2 (AR representation characterization)
X_A does not GC X_B if and only if the 21-block of $a(z)$, i.e. $a_{21}(z)$ is zero.

Theorem 3 (Spectrum based characterization)
X_A does not GC X_B if and only if $f_{AB}(z)f_{BB}^{-1}(z)$ is causal.

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GC and Additive Noise

Observations:

$$\bar{X}_A = X_A + N_A$$
$$\bar{X}_B = X_B + N_B$$

N_A, N_B stationary noise, with rational spectral densities, uncorrelated with each other and with X_A, X_B.

Questions:

- Suppose that X_A does not GC X_B, does $\bar{X}_A$ not GC $\bar{X}_B$?
- Suppose that $\bar{X}_A$ does not GC $\bar{X}_B$, can one conclude that X_A does not GC X_B ?

Note:

$$f_{\bar{A}\bar{A}} = f_{AA} + f_{N_A N_A}; \quad f_{\bar{B}\bar{B}} = f_{BB} + f_{N_B N_B}$$
$$f_{AB} = f_{\bar{A}\bar{B}}.$$

Theorem 4:

1. If $N_B = 0$ then X_A does not GC X_B if and only if $\bar{X}_A$ does not GC $\bar{X}_B$.
2. If $N_B \neq 0$ and not all unstable zeros of $f_{BB} + f_{N_B N_B}$ cancel the (unstable) zeros of f_{AB} (this "genericallyïs the case), then if X_A does not GC X_B, we have that $\bar{X}_A$ GC $\bar{X}_B$
3. It generically holds, that if $\bar{X}_A$ does not GC $\bar{X}_B$, then X_A GC X_B.

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Signal to Noise Ratio and GC

Lemma (Continuity Result)

Consider a complex matrix function $M(z)$, analytic in $\rho < |z| < \rho^{-1}; 0 < \rho < 1$ with $M(z) = M^T(z^{-1})$ and positive definite on $|z| = 1$. Suppose

$$M(z) = \sum_{i=-\infty}^{\infty} m_i z^i, \quad m_i = m_{-i}^T \in \mathbb{R}^{d \times d}$$

and define the causal and anticausal parts by

$$M_+(z) = \frac{1}{2} m_0 + \sum_{i=1}^{\infty} m_i z^i, M_-(z) = \frac{1}{2} m_0 + \sum_{i=-\infty}^{-1} m_i z^i$$

Then the matrix function $L(z) = I + \varepsilon M$ is analytic in $\rho < |z| < p^{-1}$ with $L(z) = L^T(z^{-1})$ and positive definite on $|z| = 1$.

Further, to the first order in $\varepsilon > 0$ there holds

$$L = I + \varepsilon M \approx (I + \varepsilon M_+)(I + \varepsilon M_-)$$

with $I + \varepsilon M_+$ stable and miniphase.

Signal to Noise Ratio and GC

Terminology "to first order in εïs shorthand for saying that the L_2 norm of the error Δz, defined via $\mathrm{tr}(2\pi)^{-1}\int \Delta(e^{-\iota\lambda})\Delta^*(e^{-\iota\lambda})d\lambda$ is of order ϵ: $O(\epsilon)$.
Idea: Small perturbations in a spectrum give small perturbations in a spectral factor: If X_A does not GC X_B and if the noise N_B is small, then the "violation of GC is small too".

Theorem 5 Define $\bar{X}_B = X_B + \varepsilon^{\frac{1}{2}} N_B$, so that $f_{\bar{X}\bar{X}} = f_{XX} + \varepsilon f_{NN}$ where the 11, 12, 21 blocks of f_{NN} are zero and the 22 block in $f_{N_B N_B}$. Let $W(z)Q$ with $W(z)$ upper block triangular and Q block diagonal and $\bar{W}(z), \bar{Q}$ define the canonical spectral factorizations f_{XX} and $f_{\bar{X}\bar{X}}$ respectively. Then

(i) $\bar{W}(z) - W(z)$ is $O(\varepsilon)$ on $|z| = 1$

(ii) $\bar{Q} - Q$ is $O(\varepsilon)$

(iii) $f_{A\bar{B}} f_{\bar{B}\bar{B}}^{-1} - f_{AB} f_{BB}^{-1}$ is $O(\varepsilon)$ on $|z| = 1$ and the anticausal part of $f_{A\bar{B}} f_{\bar{B}\bar{B}}^{-1}$ is $O(\varepsilon)$ on $|z| = 1$

(iv) For suitably small ε, $\bar{W}_{22}(z)$ is miniphase

So X is "close toä process in which X_A does not cause $\bar{X}_B$.

Conclusion

- Noise on X_A only does not change the "Granger causality relationöf "the A part to the B part".
- Noise on X_B generically changes the "Granger causality relationöf the "A part to the B part".
- Small noise on X_B leads to "weak" violations of Granger non causality relations

THANK YOU